concentrations of C = 0.2% (the hydrodynamic resistance drop at C = 0.2% was $\Delta\lambda/\lambda$ = 32%). The measurements showed that the tendency occurred in the mean-velocity-profile distribution for aqueous dithalan solutions as for methaupon solutions.

Thus the foregoing experiments have shown that the optical Doppler velocity meter (laser anemometer) may be used for studying turbulent flows containing surface-active additives, provided that the solutions of these are sufficiently transparent.

NOTATION

u, mean longitudinal velocity component; u_*, dynamic velocity; Re = u_*y/ν, Reynolds number; y, distance from wall; ν, kinematic viscosity; C, concentration; $\Delta\lambda/\lambda = [(\lambda_1 - \lambda_2)/\lambda_1]\cdot 100\%$, reduction in hydrodynamic resistance; λ_1, resistance for the flow of the pure solvent; λ_2, resistance for the flow of the solvent containing SAS additives; ΔP, pressure drop; h, height of channel.

LITERATURE CITED

1. I. L. Povkh, S. B. Stupin, V. M. Dobrichenko, and S. N. Maksyutenko, Inzh.-Fiz. Zh., 27, No. 3 (1974).
2. B. S. Rinkevichus, Teplofiz. Vys. Temp., 8, No. 5 (1970).
3. I. K. Nikitin, N. G. Poznyaya, and A. V. Dzevaltovskii, Hydromechanics [in Russian], Kiev (1971), p. 19.
4. E. M. Khabakhpasheva, Author's Abstract of Doctoral Dissertation, Novosibirsk (1971).

A SOLUTION METHOD FOR PROBLEMS OF FREE CONVECTIVE MOTION IN LIQUIDS

Z. P. Shul'man, É. A. Zal'tsgendler, and V. I. Baikov

UDC 536.25:532.135

The solution of the problem of free convective motion of liquids with high Prandtl and Schmidt numbers is obtained in the form of inner and outer asymptotic solutions which are joined together. Boundary conditions of the first or second kind are considered.

The interest in problems of free convection in liquids has definitely increased in recent years [1, 2]. This is due to the growing importance of liquids (Newtonian, as well as non-Newtonian) in chemical industry and power engineering. In the present article a solution method is described for the problem of free convective motion of liquids close to bodies immersed in the latter. The analysis is carried out by considering an example of a "power series" model of a liquid. In this case the dimensionless equations for stationary concentration thermally free convection in the boundary-layer approximation with the consistency coefficient dependence on heat taken into account are of the following form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\omega(\Theta_1)\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right] + (\Theta_1 + K_1\Theta_2)Mx^{\beta};$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \qquad (1)$$

$$u\frac{\partial\Theta_1}{\partial x} + v\frac{\partial\Theta_1}{\partial y} = \frac{1}{\mathrm{Pr}_1}\frac{\partial^2\Theta_1}{\partial y^2},$$

Institute of Heat and Mass Exchange, Academy of Sciences of the Belorussian SSR, Minsk.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 5, pp. 857-863, November, 1975.

$$u \frac{\partial \Theta_2}{\partial x} + v \frac{\partial \Theta_2}{\partial y} = \frac{1}{\mathrm{Pr}_2} \frac{\partial^2 \Theta_2}{\partial y^2}$$

with the boundary conditions of the first kind,

$$u = 0,\ v = 0,\ \Theta_1 = 1,\ \Theta_2 = 1 \ \text{ for } \ y = 0; \qquad u \to 0,\ \Theta_1 \to 0,\ \Theta_2 \to 0 \ \text{ for } \ y \to \infty, \tag{2}$$

where

$$K_1 = \operatorname{sign}(C_0 - C_\infty) \left(\frac{\mathrm{Gr}_2}{\mathrm{Gr}_1} \right)^{\frac{1}{2-n}},$$

and the geometric parameters M and β are in special cases as follows: M = 1, β = 0 for a vertical plate; M = cos φ, β = 0 for a wedge; M = 1, β = 1 for a plane critical point.

The solving of the system of equations (1) under the boundary conditions (2) is mathematically very difficult; in particular, (1)-(2) have no self-simulating solutions. Nevertheless, by using a physical property, namely, that the Prandtl (Pr_1) or Schmidt (Pr_2) numbers are higher, a considerable simplification of the problem is possible. In this case the thickness of the thermal and the concentration boundary layers are much smaller than the thickness of the dynamic layer, which enables one to obtain a solution by using a method of asymptotic expansions joined together.

By changing over to inner variables one has for the inner asymptotic expansion of the system of equations (1)

$$\begin{gathered} \frac{\partial}{\partial y_1} \left[\omega(\Theta_1) \left| \frac{\partial u_1}{\partial y_1} \right|^{n-1} \frac{\partial u_1}{\partial y_1} \right] + (\Theta_1 + K_1 \Theta_2) M x^\beta = 0; \\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y_1} = 0; \\ u_1 \frac{\partial \Theta_1}{\partial x} + v_1 \frac{\partial \Theta_1}{\partial y_1} = \frac{\mathrm{Pr}_i}{\mathrm{Pr}_1} \frac{\partial^2 \Theta_1}{\partial y_1^2}; \\ u_1 \frac{\partial \Theta_2}{\partial x} + v_1 \frac{\partial \Theta_2}{\partial y_1} = \frac{\mathrm{Pr}_i}{\mathrm{Pr}_2} \frac{\partial^2 \Theta_2}{\partial y_1^2}, \end{gathered} \tag{3}$$

where

$$\mathrm{Pr}_i = \min\{\mathrm{Pr}_1,\ \mathrm{Pr}_2\}.$$

The problems of free convection also reduce to the system (3) close to the vertical cylinder (M = 1, β = 0) or to the cone (M = $[2 \sin \varphi]^{n/2} \cos \varphi$, β = n/2) of the critical point in space [M = $2^{(n+1/2)}$, β = (n + 1)/2]. One also has the continuity equation

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0$$

and, in addition, one introduces

$$\bar{u} = ur, \quad \bar{x} = \int_0^x r(x)\,dx.$$

The condition u → 0 for y → ∞ in the case of inner asymptotic expansion must be replaced by a condition that u_1 is bounded or, in view of physical considerations, by an equivalent condition $\partial u_1/\partial y_1 \to 0$. Thus the boundary conditions for the system (3) are

$$u_1 = 0,\ v_1 = 0,\ \Theta_1 = 1,\ \Theta_2 = 1 \ \text{ for } \ y_1 = 0, \qquad \frac{\partial u_1}{\partial y_1} \to 0,\ \Theta_1 \to 0,\ \Theta_2 \to 0 \ \text{ for } \ y_1 \to \infty. \tag{4}$$

In the case of the outer asymptotic expansion (Θ_1 = 0, Θ_2 = 0) one arrives at the following system of equations (in outer variables):

$$u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y_2} = \frac{\partial}{\partial y_2}\left[\left|\frac{\partial u_2}{\partial y_2}\right|^{n-1} \frac{\partial u_2}{\partial y_2}\right];$$
$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y_2} = 0. \quad (5)$$

In this case the condition of adhesion on the surface cannot be satisfied. Another condition on the surface is found by using the "principle of joining together in the limit" [4]. Then the boundary conditions for the system (5) become

$$u_2 = U, \ v_2 = 0 \ \text{for} \ y_2 = 0; \ u_2 \to 0 \ \text{for} \ y_2 \to \infty; \quad (6)$$

where

$$U = \lim_{y_1 \to \infty} u_1(x, y_1).$$

The representation of the problem (1), (2) in the form of inner (3), (4) asymptotic expansion and of outer (5), (6) greatly simplifies the problem. In particular, the problem (3), (4) admits self-simulating solutions in contrast to the original problem (4). One has the following system of differential equations:

$$\frac{d}{d\eta}[\omega(\Theta_1)|f_1''|^{n-1} f_1''] + \Theta_1 + K_1\Theta_2 = 0;$$
$$\Theta_1'' + \frac{\Pr_1}{\Pr_i}(\operatorname{sign} f_1')f_1\Theta_1' = 0; \quad (7)$$
$$\Theta_2'' + \frac{\Pr_2}{\Pr_i}(\operatorname{sign} f_1')f_1\Theta_2' = 0$$

with the boundary conditions

$$f_1 = 0, \ f_1' = 0, \ \Theta_1 = 1, \ \Theta_2 = 1 \ \text{for} \ \eta_1 = 0;$$
$$f_1'' \to 0, \ \Theta_1 \to 0, \ \Theta_2 \to 0 \ \text{for} \ \eta_1 \to \infty. \quad (8)$$

An exact solution of the problem (7)-(8) was obtained in [4]. The inner asymptotic expansion enables one to find the characteristics of the heat and mass exchange and the surface friction, as well as the temperature and the concentration profiles. However, to obtain the complete profiles of the velocities one must also find the outer asymptotic expansion. It follows from the "principle of joining together in the limit" and from the obtained solution for the inner asymptotic expansion that

$$U = f_1'(\infty)\Pr_i^{\frac{n+1}{3n+1}} x^{\frac{n+1+2\beta}{3n+1}} M^{\frac{2}{3n+1}}\left(\frac{3n+1}{2n+1+\beta}\right)^{\frac{n+1}{3n+1}}. \quad (9)$$

The going over in (5) and (6) to flow functions, the requirement of constant conformal invariance of the obtained system of equations relative to the linear one-parameter transformation group [5], and the use of (9), yield the self-simulating variables,

$$\eta_2 = C_1 y_2 x^{-\frac{n^2+2n-1-2\beta(2-n)}{(3n+1)(n+1)}};$$
$$f_2(\eta_2) = C_2 \Psi_2 x^{-\frac{2[n(n+2)+\beta(2n-1)]}{(3n+1)(n+1)}}, \quad (10)$$

where

$$C_1 = D_1^{\frac{2-n}{n+1}}; \ C_2 = D_1^{\frac{1-2n}{n+1}}; \ D_1 = f_1'(\infty)\Pr_i^{\frac{n+1}{3n+1}} M^{\frac{2}{3n+1}}\left(\frac{3n+1}{2n+1+\beta}\right)^{\frac{n+1}{3n+1}},$$

which reduce the problem (5),(6) to a single nonlinear differential equation:

$$n|f_2''|^{n-1} f_2''' - \frac{n+1+2\beta}{3n+1}(f_2')^2 + \frac{2[n(n+2)+\beta(2n-1)]}{(3n+1)(n+1)} f_2 f_2'' = 0 \quad (11)$$

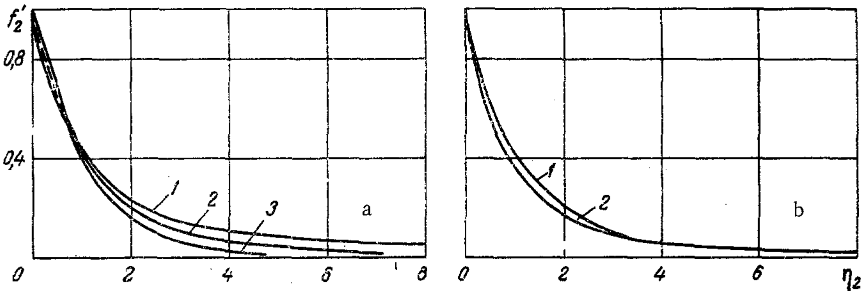

Fig. 1. Profiles of velocities for the outer asymptotic expansion: a: 1) n = 0.5; 2) n = 0.75; 3) n = 1.0 (β = 0); b: 1) β = 0; 2) β = 1 (n = 0.75).

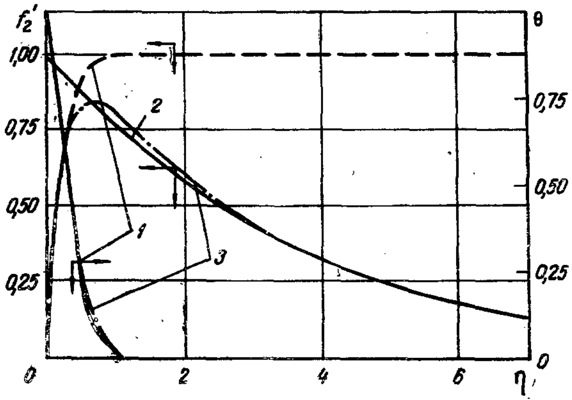

Fig. 2. Profiles of temperatures (concentrations) and of velocities: 1) inner asymptotic expansion; 2) outer asymptotic expansion; 3) exact solution.

with the boundary conditions

$$f_2(0) = 0, \quad f_2'(0) = 1, \quad f_2'(\infty) \to 0. \tag{12}$$

The problem (11), (12) was solved numerically on the "Minsk-22" electronic computer. Some results are shown in Fig. 1. An increase in the pseudoplastic properties (a reduction in the parameter of non-Newtonian behavior) results in greater thickness of the dynamic boundary layer in self-simulating variables (Fig. 1a). The geometric parameter β shows hardly any effect on the velocity profiles (Fig. 1b).

In Fig. 2 the profiles of temperatures (concentrations) are shown, as well as velocities found by using the above-described procedure; they were determined by solving the complete problem (1), (2) for $n = 1$, $Pr_1 = Pr_2 = 100$, $K_1 = 0$, $M = 1$, $\beta = 0$ [for $n = 1$, the problem (1), (2) admits self-simulating solutions]. It can be seen from the graphs that the results agree very satisfactorily; for example, the difference between the characteristics of heat and mass exchange and of friction nowhere exceeds 3.5% (the constituent profile of the velocities is obtained from the intervals of profiles of the inner and outer asymptotic expansions up to their point of intersection). For higher numbers Pr_k ($k = 1, 2$) the error in the determination of both the characteristics on the surface and the temperature profiles, concentrations and velocities is reduced.

It should be mentioned that the solution of the problem (1), (2) is of universal character as regards the Prandtl and Schmidt numbers, The Grashof numbers (Gr_1, Gr_2), and the shape of the heat-dependent function $\omega(\Theta_1)$.

In the case of electrical heating of the surfaces and in a number of other cases of thermal convection it is not the surface temperature which is known or given but the heat flow, that is, the boundary conditions of the second kind are realized. In this case one can also use the method of fitted asymptotic expansions. For the internal asymptotic expansion in the case of a constant consistency coefficient the system of equations is of the form (3) [if one sets $\Theta_2 = 0$, $Pr_i = Pr_1$, $\omega(\Theta_1) = 1$] with the boundary conditions

$$u_1 = 0,\ v_1 = 0,\ \frac{\partial \Theta_1}{\partial y_1} = -1 \ \text{for}\ \ y_1 = 0; \qquad (13)$$

$$\frac{\partial u_1}{\partial y_1} \to 0,\ \Theta_1 \to 0 \ \text{for}\ \ y_1 \to \infty,$$

where

$$y = \frac{y'}{L} \mathrm{Gr}_1^{\frac{1}{n+4}} \mathrm{Pr}_1^{\frac{n}{3n+2}}; \quad u_1 = u' \left[\frac{g\beta_1 L^2 q_0}{\lambda}\right]^{-\frac{1}{2}} \mathrm{Gr}_1^{\frac{1}{2(n+4)}} \mathrm{Pr}_1^{\frac{n+2}{3n+2}};$$

$$\Theta_1 = \frac{(T - T_\infty)\lambda}{q_0 L} \mathrm{Gr}_1^{\frac{1}{n+4}} \mathrm{Pr}_1^{\frac{n}{3n+2}}; \quad v_1 = v' \left[\frac{g\beta_1 L^2 q_0}{\lambda}\right]^{-\frac{1}{2}} \mathrm{Gr}_1^{\frac{3}{2(n+4)}} \mathrm{Pr}_1^{\frac{2(n+1)}{3n+2}}$$

For the external asymptotic expansion the equations are again of the form (5) with the boundary conditions (6). The internal asymptotic expansion possesses a self-simulating solution [6]: the system of ordinary differential equations

$$n|f_1''|^{n-1} f_1''' + g = 0;$$
$$g'' + f_1 g' - \frac{n - \beta}{2(n+1) + \beta} f_1' g = 0 \qquad (14)$$

and the boundary conditions

$$f_1 = 0,\ f_1' = 0,\ g' = -1 \ \text{for}\ \eta_1 = 0;\ f_1'' \to 0,\ g \to 0 \ \text{for}\ \eta_1 \to \infty. \qquad (15)$$

The numerical solution of the problem (14), (15) was obtained by using a modified Newton's method described in [6]. In this case the outer limit for the inner asymptotic expansion is given by

$$U = f_1'(\infty) \mathrm{Pr}_1^{\frac{n+2}{3n+2}} x^{\frac{n+2(1+\beta)}{3n+2}} M^{\frac{2}{3n+2}} \left[\frac{3n+2}{2(n+1)+\beta}\right]^{\frac{n+2}{3n+2}}.$$

The proceeding to stream functions in (5) and (6), as well as the requirement of constant conformal invariance of the obtained system of equations relative to a linear one-parameter transformation group, results in the self-simulating variables

$$\eta_2 = C_3 y_2 x^{-\frac{n^2+3n-2-2\beta(2-n)}{(3n+2)(n+1)}};$$
$$f_2(\eta_2) = C_4 \Psi_2 x^{-\frac{2[n(n+3)+\beta(2n-1)]}{(3n+2)(n+1)}}, \qquad (16)$$

where

$$C_3 = D_2^{\frac{2-n}{n+1}};\ C_4 = D_2^{\frac{1-2n}{n+1}};\ D_2 = f_1'(\infty) \mathrm{Pr}_1^{\frac{n+2}{3n+2}} M^{\frac{2}{3n+2}} \left[\frac{3n+2}{2(n+1)+\beta}\right]^{\frac{n+2}{3n+2}},$$

which reduce the problem of (5), (6) to a nonlinear differential equation:

$$n|f_2''|^{n-1} f_2''' - \frac{n+2(1+\beta)}{3n+2}(f_2')^2 + \frac{2[n(n+3)+\beta(2n-1)]}{(n+1)(3n+2)} f_2 f_2'' = 0 \qquad (17)$$

with (12) as boundary conditions.

In Fig. 3 some results are shown of the numerical solution of the differential equation (17) subject to boundary conditions (12). The parameter of non-Newtonian behavior and the geometric factor exercise in qualitative terms the same effect whether one deals with boundary conditions of the first or the second kind.

It is noted, in conclusion, that by representing the solution by inner and outer asymptotic expansions joined together one is also able to construct efficient approximate solutions

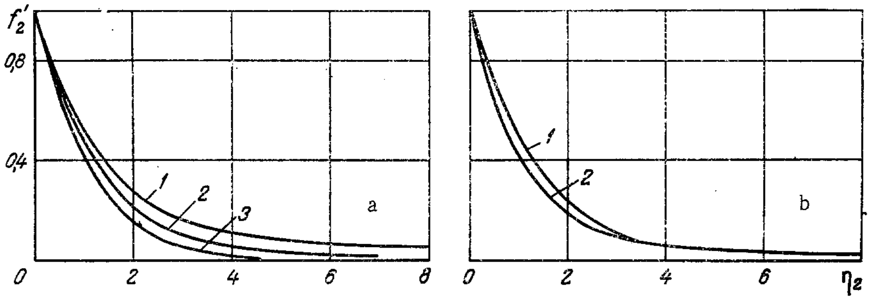

Fig. 3. Velocity profiles for the outer asymptotic expansion: a: 1) n = 0.5; 2) n = 0.75; 3) n = 1.0 (β = 0); b: 1) β = 0; 2) β = 1 (n = 0.75).

NOTATION

$x = x'/L$; $y = (y'/L)Gr_1^{1/2(n+1)}$; $y_1 = yPr_1^{n/(3n+1)}$; $y_2 = y$, dimensionless coordinates; x', y', dimensional coordinates; $u = u'[Lg\,\beta_1(T_0 - T_\infty)]^{-1/2}$, $v = v'[Lg\,\beta_1(T_0 - T_\infty)]^{-1/2}\cdot Gr_1^{1/2(n+1)}$, $u_1 = uPr_1^{(n+1)/(3n+1)}$, $u_2 = u$, $v_1 = vPr_1^{(2n+1)/(3n+1)}$, $v_2 = v$, dimensionless velocities; u', v', dimensional velocities; L, characteristic length; β_1, coefficient of volume expansion; $Pr_1 = (\rho c_p/\lambda)L^{(1-n)/(n+1)}(\rho/k)^{-2/(n+1)}[L\beta_1 g(T_0 - T_\infty)]^{3(n-1)/2(n+1)}$, $Pr_2 = (\rho c_p/D)(\rho/k)^{-[2/(n+1)](1-n)/(n+1)}[L\beta_1 g(T_0 - T_\infty)]^{3(n-1)/2(n+1)}$, modified Prandtl and Schmidt numbers, respectively; $Gr_1 = (\rho/k)^2L^{n+2}[\beta_1 g(T_0 - T_\infty)]^{2-n}$, $Gr_2 = (\rho/k)^2L^{n+2}[\beta_2 g|C_0 - C_\infty|]^{2-n}$, modified Grashof numbers; k, consistency coefficient; n, parameter of non-Newtonian behavior; Ψ, stream function; $\omega(\Theta_1)$, heat-dependent function; $\Theta_1 = (T - T_\infty)/(T_0 - T_\infty)$, $\Theta_2 = (C - C_\infty)/(C_0 - C_\infty)$, dimensionless temperature and concentration, respectively; T_0, C_0, T_∞, C_∞, absolute temperatures and concentrations on the wall and for $y \to \infty$; q_0, heat flow on the surface; η, f(η), g(η), self-simulating variables.

LITERATURE CITED

1. A. D. Id, in: Achievements in Heat Transfer [Russian translation], Mir (1970).
2. A. Emery, K. Tsi, and I. Dale, Teploperedacha, No. 2 (1971).
3. M. Van-Dyke, Perturbation Methods in Fluid Mechanics, Academic Press (1964).
4. Z. P. Shul'man, É. A. Zal'tsgendler, and V. I. Baikov, Inzh.-Fiz. Zh., 27, No. 3 (1974).
5. S. G. Lee and W. F. Ames, Amer. Inst. Chem. Eng. J., 12, No. 4 (1966).
6. Z. P. Shul'man, É. A. Zal'tsgendler, and V. I. Baikov, Dokl. Akad. Nauk BelorusSSR, 19, No. 1 (1975).